

## OPTIMAL DESIGN OF RIB-STIFFENERS IN DISKS AND PLATES

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**Abstract**—Disks and plates subjected to stretching and flexure are stiffened by curvilinear rib-stiffeners whose shape and cross-sectional area are subject to optimization. Sensitivity analysis is first discussed for an arbitrary integral functional of generalized stress, strain and displacement and boundary domains. The optimality conditions are next derived. The general theory is illustrated by three examples of optimal design of ring-stiffeners in circular disks and plates and of rib-stiffeners in a rectangular plate.

### I. INTRODUCTION

In the present work, disks and plates subjected to stretching and flexure are considered within the assumptions of small strain theory. The stiffening ribs are assumed to be introduced within the disk or plate domain in order to reduce deflections or stresses. The shape of such interior ribs and their cross-sectional areas may vary and sensitivity of any response functional with respect to this variation is to be determined. Next, the optimal design problem is considered for which the optimal shape, position and cross-sectional stiffness of the rib is sought in order to minimize the objective functional.

The present formulation constitutes an extension of a class of problems for which an optimal point support reaction on structures was to be determined, cf. Mróz (1980, 1987). Instead of point action, a stiffener exerts a line action inducing traction discontinuity along the rib. This discontinuity is related to forces transformed by the rib through equilibrium conditions. The variation of rib shape will thus correspond to variation of a line of traction discontinuity within the plate domain. The class of problems of sensitivity analysis with discontinuity lines was briefly discussed by Mróz (1986) where three types of discontinuity surfaces were introduced. In this paper, a detailed analysis will be provided for both stretching and flexure of rib-stiffened plates for which both sensitivity expressions and optimality conditions will be derived. Though stability and vibration problems are not treated here, the analysis can easily be extended to these cases, as it has been done by Dems and Mróz (1989b).

In general, the design sensitivity of any arbitrary functional specified over structure domain can be obtained by two approaches. The direct sensitivity method requires an additional solution of boundary-value problem for each particular variation of design variable or parameter. The adjoint state method requires only one additional solution of an adjoint problem for specified functional independently of design variations. These two approaches have received considerable attention in the literature, cf. Choi and Haug (1983), Dems and Mróz (1984), Choi and Seong (1986), Haber (1986), Mróz (1986), Dems and Haftka (1989) and others. The choice between these approaches depends on number of

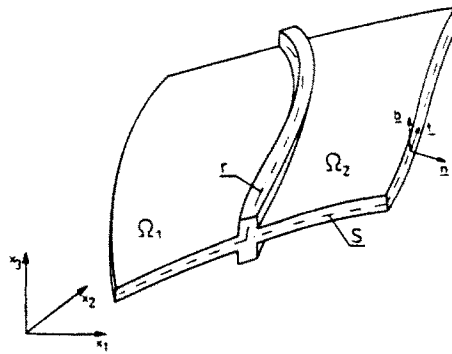


Fig. 1. Thin plate stiffened by a rib.

objective functionals and design variations, and also on the relative difficulty of obtaining adjoint or direct solutions. In this paper, only one objective functional is considered and therefore the adjoint state method will be applied to derive sensitivity expressions for arbitrary variations of stiffener cross-section and shape. In this sense, the present analysis constitutes an extension of previous works by Dems and Mróz (1984, 1987, 1989a).

In Section 2, the equivalent model of a stiffened plate will be presented and in Sections 3 and 4 the sensitivity analysis for disks and plates will be carried out. The optimality conditions are generated next from sensitivity expressions in Section 5. Several illustrative examples are presented in Section 6.

2. EQUIVALENT MODEL OF A STIFFENED PLATE

Following Washizu (1975), consider a thin plate with the plane middle surface and introduce a fixed Cartesian reference system with the  $x_1, x_2$ -axes lying within the middle surface and the  $x_3$ -axis being directed normally to the middle surface. The transverse boundary surfaces of the plate are cylindrical and parallel to the  $x_3$ -axis.

Assume the plate to be stiffened by a rib of an arbitrary shape and a cross-section symmetrical with respect to the middle surface (Fig. 1). Denote the plate domain by  $\Omega = \Omega_1 \cup \Omega_2$ , where  $\Omega_1$  and  $\Omega_2$  are sub-domains specified by a rib intersecting the boundary  $S$ . The rib axis  $\Gamma$  lies within  $\Omega$  and may have its end points on  $S$ . For convenience of the subsequent analysis, introduce a local right-handed rectangular coordinate system  $\mathbf{n}, \mathbf{t}, \mathbf{b}$  along  $S$  or  $\Gamma$ , such that  $\mathbf{n}$  is vector normal to  $S$  or  $\Gamma$  pointing in the exterior of  $\Omega$  or into  $\Omega_2$ ,  $\mathbf{t}$  is tangential to  $S$  or  $\Gamma$ , and  $\mathbf{b}$  is normal to  $\Omega$ .

The plate can be subjected to a distributed lateral pressure  $p$  per unit area of the middle surface, directed along the  $x_3$ -axis, and to body force  $\mathbf{f}$  acting within the  $x_1, x_2$ -plane. On a portion of the plate boundary  $S_r$ , generalized forces per unit length of the plate boundary are specified. With respect to the local system  $\mathbf{n}, \mathbf{t}, \mathbf{b}$ , we have, cf. Fig. 2a,

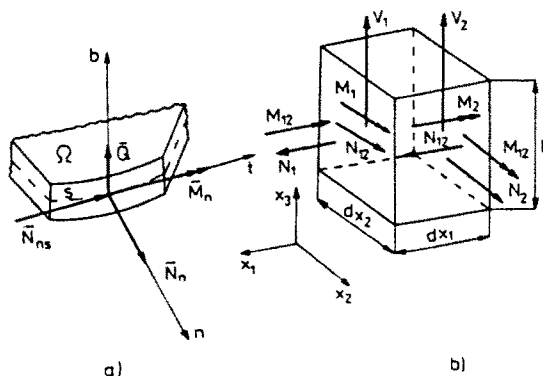


Fig. 2. Generalized boundary tractions (a) and stresses (b) for a plate.

$$\tilde{\mathbf{T}} = \begin{Bmatrix} \tilde{N}_{ns} \\ \tilde{N}_n \end{Bmatrix}, \quad \tilde{\mathbf{R}} = \begin{Bmatrix} \tilde{Q} \\ \tilde{M}_n \end{Bmatrix} \quad \text{on } S_T \tag{1}$$

where  $\tilde{N}_{ns}$  and  $\tilde{N}_n$  are the in-plate tractions acting along the  $\mathbf{t}$ - and  $\mathbf{n}$ -axes,  $\tilde{Q} = \tilde{V} + \tilde{M}_{ns}$ , denotes the generalized shear force acting along the  $\mathbf{b}$ -axis, while  $\tilde{M}_n$  is the normal bending moment acting along  $S$ . The twisting moment and the boundary shear force are denoted by  $\tilde{M}_{ns}$  and  $\tilde{V}$ . On the remaining portion  $S_u$  of the boundary  $S$ , the homogeneous displacement conditions are imposed, namely

$$\tilde{\mathbf{u}} = \begin{Bmatrix} \tilde{u}_t \\ \tilde{u}_n \end{Bmatrix} = 0, \quad \tilde{\mathbf{w}} = \begin{Bmatrix} \tilde{w} \\ \tilde{w}_{,n} \end{Bmatrix} = 0 \quad \text{on } S_u \tag{2}$$

where  $\tilde{u}_t, \tilde{u}_n$  are the in-plane displacements in the directions of the  $t, n$ -axes, and  $w$  denotes the deflection of the middle surface in the  $x_3$ -direction. The comma preceding an index denotes partial differentiation in local or global reference frames.

Note that for a linear case, the body forces  $\mathbf{f}$  and in-plane forces  $\tilde{N}_{ns}, \tilde{N}_n$  yield the stretching of the plate. On the other hand, the lateral pressure  $p$  with generalized boundary forces  $\tilde{Q}, \tilde{M}_n$  yield plate flexure. These two modes of deformation are not coupled and can be treated separately. The generalized stresses in stretching and bending per unit length of  $x_1$  and  $x_2$  lines are defined as follows

$$\mathbf{Q} = \begin{Bmatrix} N_1 \\ N_2 \\ N_{12} \end{Bmatrix}, \quad \mathbf{M} = \begin{Bmatrix} M_1 \\ M_2 \\ M_{12} \end{Bmatrix} \tag{3}$$

where  $N_1, N_2, N_{12}$  are the in-plane stress resultants and  $M_1, M_2, M_{12}$  are the bending and twisting moments. The shearing forces are denoted by  $V_1$  and  $V_2$ . The corresponding generalized strains are

$$\mathbf{q} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix}, \quad \boldsymbol{\chi} = \begin{Bmatrix} \chi_1 \\ \chi_2 \\ \chi_{12} \end{Bmatrix} = - \begin{Bmatrix} w_{,11} \\ w_{,22} \\ w_{,12} \end{Bmatrix} \tag{4}$$

where  $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$  are the in-plane strains and  $\chi_1, \chi_2, \chi_{12}$  denote the curvatures and torsion of the middle surface.

Introducing a rib along the line  $\Gamma$ , the field of internal forces is redistributed so that discontinuities occur along  $\Gamma$ . This redistribution will be treated within the plate theory, thus neglecting local effects near the rib. The rib is treated as a plane curvilinear arch with free or supported ends  $A$  and  $B$  subject to stretching, bending and torsion due to the external loading corresponding to discontinuities of internal forces within the plate along the curve  $\Gamma$ . This loading is related to the generalized stresses within the arch through the equilibrium

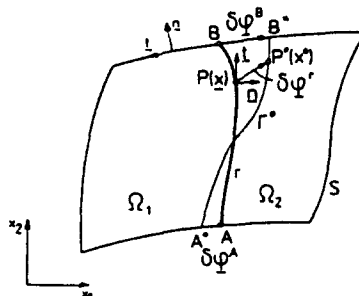


Fig. 3. Model of a stiffened plate.

equations. The generalized displacements of the arch are the same as the respective plate displacements along  $\Gamma$ . The plate stiffened by a rib is thus approximated by two domains  $\Omega_1$  and  $\Omega_2$  separated by the line  $\Gamma$  of strong discontinuity of generalized stresses (Fig. 3).

The boundary value problem can be solved for a specified geometry, loading and material properties of a plate. Consider now a more complex problem when the cross-sectional properties and the shape of rib are not specified in advance. The major question can be posed as how generalized stresses, strains and displacements or some global functionals are modified due to cross-section and shape variations of the rib. Assume the cross-sectional property and the shape to be dependent on a set of material design functions  $\gamma_l(\mathbf{x})$ , ( $l = 1, 2, \dots, L$ ),  $\mathbf{x} \in \Gamma$ , and shape design functions  $\phi_k(\mathbf{x})$ , ( $k = 1, 2$ ),  $\mathbf{x} \in \Gamma$ . Our purpose is to express explicitly the variation of an arbitrary functional in terms of variations of  $\gamma_l(\mathbf{x})$  and  $\phi_k(\mathbf{x})$ , cf. Fig. 3. It is also assumed that the plate domain may undergo the infinitesimal transformation  $\delta\phi(\Omega)$  where  $\delta\phi$  is a differentiable vector field satisfying the conditions

$$\begin{aligned} \delta\phi(\Omega) &= \delta\phi(\Gamma) \quad \text{for } \mathbf{x} \in \Gamma \\ \delta\phi \cdot \mathbf{n} &= 0 \quad \text{for } \mathbf{x} \in S. \end{aligned} \tag{5}$$

Thus the external boundary does not undergo any normal shape transformation. On the other hand, when the rib penetrates the external plate boundary, cf. Fig. 3, the tangential shape transformation may occur. The additional constraint on the vector field  $\delta\phi$  at points A and B of  $\Gamma$  is then to be imposed, namely

$$\delta\phi_t + \delta\phi_n \operatorname{ctg} \alpha = 0 \quad \text{at A or B} \tag{6}$$

where  $\delta\phi_t$  and  $\delta\phi_n$  are the tangential and normal components of rib variation, and  $\alpha$  denotes the angle between the rib and the external plate boundary. When  $\alpha$  does not vary during the transformation of rib shape, the following condition must be satisfied at A and B:

$$\delta\phi_{n,s} = \left( K^S \frac{1 - 2 \sin^2 \alpha}{\sin \alpha} - K^\Gamma \right) \delta\phi_n \quad \text{at A or B} \tag{7}$$

where  $K^S$  and  $K^\Gamma$  denote the curvatures of  $S$  and  $\Gamma$  at A or B, respectively.

Due to an infinitesimal transformation of rib shape, variation of orientation of the local coordinate system  $\mathbf{n}$ ,  $\mathbf{t}$ ,  $\mathbf{b}$ , and of rib length and its curvature occurs, that is (cf. Dems and Mróz, 1987)

$$\begin{aligned} \overset{\circ}{\delta}\mathbf{t} &= \mathbf{n}(K\delta\phi_t + \delta\phi_{n,s}), \quad \overset{\circ}{\delta}\mathbf{n} = -\mathbf{t}(K\delta\phi_t + \delta\phi_{n,s}), \quad \overset{\circ}{\delta}\mathbf{b} = 0 \\ \delta(d\Gamma) &= (\delta\phi_{s,ss} - K\delta\phi_n) d\Gamma, \quad \delta K = K_s\delta\phi_s + K^2\delta\phi_n + \delta\phi_{n,sss} \end{aligned} \tag{8}$$

where  $s$  denotes the rib parameter,  $\overset{\circ}{\delta}$  denotes the total variation of any vector quantity, and  $\delta$  denotes the variation of any scalar quantity.

### 3. SENSITIVITY ANALYSIS FOR DISKS WITH VARYING STIFFENING RIB

In this section, we shall consider a disk or plate subject to stretching in its plane. The plate is loaded by the body forces  $\mathbf{f}$  within its domain, by surface tractions  $\overset{\circ}{\mathbf{T}}$  on  $S_r$  and with vanishing displacement  $\overset{\circ}{\mathbf{u}} = 0$  on  $S_u$ . Denote by  $\mathbf{v}$ ,  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\Sigma}$  the displacement, generalized strain and stress field along the rib axis, where

$$\mathbf{v} = \begin{Bmatrix} v_t \\ v_n \end{Bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon \\ \varepsilon_r \end{Bmatrix}, \quad \boldsymbol{\Sigma} = \begin{Bmatrix} N \\ M \end{Bmatrix}. \tag{9}$$

Here  $v_t$ ,  $v_n$  denote the tangential and normal components of the displacement vector on  $\Gamma$ ,

$\varepsilon$  and  $\chi$  are the elongation and curvature of the rib element and  $N$ ,  $M$  denote the cross-sectional normal force and bending moment. The relation between the generalized strains and displacements has the form

$$\varepsilon = v_{,ss} - Kv_{,s}, \quad \chi = -\theta_{,s} = -(v_{,ns} + Kv_{,s})_{,s} \tag{10}$$

where  $K$  denotes the rib curvature,  $\theta = v_{,ns} + Kv_{,s}$  is the angle of rotation of the rib cross-section and  $s$  denotes the arch parameter.

The displacements and generalized strains and stresses within subdomains  $\Omega_1$  and  $\Omega_2$  of the plate are denoted by  $\mathbf{u}^1$ ,  $\mathbf{u}^2$ ,  $\mathbf{q}^1$ ,  $\mathbf{q}^2$  and  $\mathbf{Q}^1$ ,  $\mathbf{Q}^2$ , respectively. It is obvious that the following kinematic relations between the displacements and strains of the plate and rib hold along  $\Gamma$ :

$$\mathbf{u}^1 = \mathbf{u}^2 = \mathbf{v}, \quad \varepsilon_s^1 = \varepsilon_s^2 = \varepsilon \quad \text{on } \Gamma \tag{11}$$

where  $\varepsilon_s^i$  denotes the strain component in the direction tangential to  $\Gamma$  within subdomain  $\Omega_i$  of plate. The generalized stresses  $\Sigma$  within a rib are related to the jump of internal forces within plate domain along  $\Gamma$  by the equilibrium equations

$$\begin{aligned} N_{,s} - M_{,ss} - K - \llbracket N_{ns} \rrbracket &= 0 \\ NK + M_{,ss} - \llbracket N_n \rrbracket &= 0 \end{aligned} \tag{12}$$

where  $\llbracket N_{ns} \rrbracket$  and  $\llbracket N_n \rrbracket$  denote the jumps of tangential and normal components of internal forces calculated as the difference of respective components in subdomains  $\Omega_1$  and  $\Omega_2$  along  $\Gamma$  (see Fig. 4a), and  $M_{,ss} = Q$  can be regarded as the cross-sectional shear force within a rib.

It is now assumed that generalized stress-strain relations within plate and rib domains are given in the general non-linear form

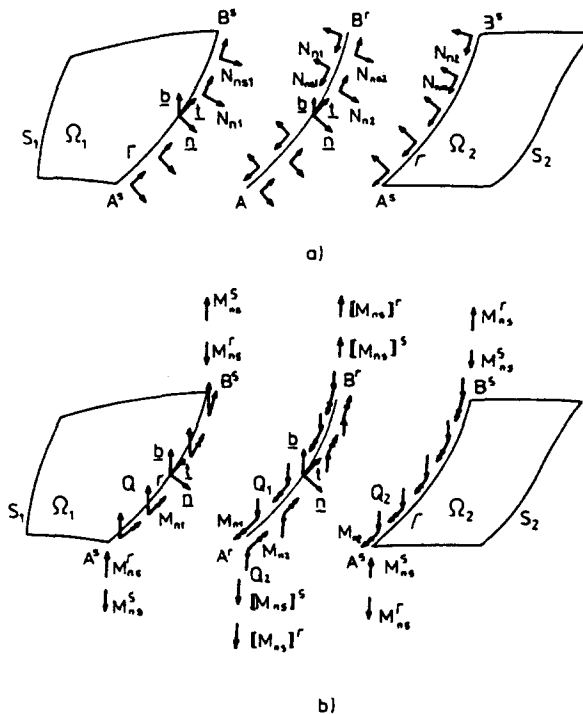


Fig. 4. Generalized forces acting along rib for stretching (a) and bending (b) of a plate.

$$\mathbf{Q} = \mathbf{S}(\mathbf{q}), \quad \boldsymbol{\Sigma} = \boldsymbol{\Lambda}(\boldsymbol{\varepsilon}, \gamma). \quad (13)$$

In the case of elasticity,  $\mathbf{S}$  and  $\boldsymbol{\Lambda}$  are generated by a potential rule associated with the specific strain energies. The incremental form of eqn (13) is expressed as follows

$$\delta\mathbf{Q} = \mathbf{D} \cdot \delta\mathbf{q}, \quad \delta\boldsymbol{\Sigma} = \mathbf{E} \cdot \delta\boldsymbol{\varepsilon} + \mathbf{L}^l \delta\gamma_i \quad (14)$$

where

$$\mathbf{D} = \partial\mathbf{S}/\partial\mathbf{q}, \quad \mathbf{E} = \partial\boldsymbol{\Lambda}/\partial\boldsymbol{\varepsilon}, \quad \mathbf{L}^l = \partial\boldsymbol{\Lambda}/\partial\gamma_i. \quad (15)$$

For a stable elastic material,  $\mathbf{D}$  and  $\mathbf{E}$  are symmetric and positive definite tangent stiffness matrices, whereas  $\mathbf{L}^l$  represents the variation of stress within the rib due to the variation of material design functions  $\gamma_i$ .

### 3.1. Sensitivity analysis for an arbitrary functional

Let us now consider the functional

$$G = \int \Psi(\mathbf{Q}, \mathbf{q}, \mathbf{f}, \mathbf{u}) \, d\Omega + \int h(\mathbf{T}, \mathbf{u}) \, dS + \int \Phi(\boldsymbol{\Sigma}, \boldsymbol{\varepsilon}, \gamma) \, d\Gamma \quad (16)$$

where the function  $h$  depends only on the local components of surface tractions  $\mathbf{T}$  and displacements  $\mathbf{u}$  on  $S$ , i.e.  $h = h(N_n, N_n, u_n, u_n)$ . We assume furthermore that  $\Psi$ ,  $h$  and  $\Phi$  are continuous functions of their arguments, and domains of integration  $\Omega$ ,  $S$  and  $\Gamma$  depend on transformation vector field  $\boldsymbol{\phi}(\mathbf{x})$  associated with modification of the shape of the rib. The variation  $\delta\boldsymbol{\phi}(\mathbf{x})$  satisfies the conditions (5) and (6).

Rewriting the functional (16) in the form

$$G = G_1 + G_2 + G_r = \left( \int \Psi \, d\Omega_1 + \int h \, dS_1 \right) + \left( \int \Psi \, d\Omega_2 + \int h \, dS_2 \right) + \int \Phi \, d\Gamma \quad (17)$$

where  $\Omega = \Omega_1 \cup \Omega_2$  and  $S = S_1 \cup S_2$ , we note that the functionals  $G_1$  and  $G_2$  are defined over domains  $\Omega_1$  and  $\Omega_2$  with varying boundary parts  $\Gamma$ , whereas the functional  $G_r$  is defined along plane arch  $\Gamma$  of varying shape and material properties (see Fig. 1a). To derive the first variation of functional  $G$  we shall follow the analysis presented by Dems and Mróz (1984, 1987) and Mróz *et al.* (1985). Following then Dems and Mróz (1984, 1987), we introduce the adjoint stiffened plate of the same shape as the primary one, but satisfying the boundary conditions

$$\tilde{\mathbf{T}}^a = h_{,u} \quad \text{on } S_r, \quad \tilde{\mathbf{u}}^a = -h_{,T} \quad \text{on } S_u \quad (18)$$

where  $S_r$  denotes the portion of  $S$  where  $\tilde{\mathbf{T}}$  is specified for the primary plate, while  $S_u$  is the remaining portion of  $S$  where  $\tilde{\mathbf{u}}$  is prescribed. The adjoint plate is subject to imposed fields of body force and initial strain and stress specified by

$$\begin{aligned} \mathbf{f}^a &= \Psi_{,u}, & \mathbf{q}^a &= \Psi_{,Q}, & \mathbf{Q}^a &= \Psi_{,q} & \text{within } \Omega_1 \cup \Omega_2 \\ \boldsymbol{\varepsilon}^a &= \Phi_{,\boldsymbol{\varepsilon}}, & \boldsymbol{\Sigma}^a &= \Phi_{,\boldsymbol{\Sigma}} & \text{along } \Gamma. \end{aligned} \quad (19)$$

The stress field  $\mathbf{Q}^a$  and  $\boldsymbol{\Sigma}^a$  within plate and rib domains are related to strain fields  $\mathbf{q}^a$  and  $\boldsymbol{\varepsilon}^a$  by the relations

$$\begin{aligned} \mathbf{Q}^a &= \mathbf{D}^T \cdot (\mathbf{q}^a - \mathbf{q}^a) - \mathbf{Q}^a & \text{within } \Omega_1 \cup \Omega_2 \\ \boldsymbol{\Sigma}^a &= \mathbf{E}^T \cdot (\boldsymbol{\varepsilon}^a - \boldsymbol{\varepsilon}^a) - \boldsymbol{\Sigma}^a & \text{along } \Gamma \end{aligned} \quad (20)$$

where  $\mathbf{D}^T$ ,  $\mathbf{E}^T$  are the transposed stiffness matrices specified by eqns (15), and  $\mathbf{Q}^a$ ,  $\Sigma^a$  satisfy the equilibrium conditions, whereas  $\mathbf{q}^a$ ,  $\boldsymbol{\varepsilon}^a$  are the associated strain fields which follow from the displacement fields  $\mathbf{u}^a$ ,  $\mathbf{v}^a$ . The jump of internal forces within the adjoint plate along  $\Gamma$  is related to the generalized stresses  $\Sigma^a$  by the equations similar to (12), while the kinematic relations along  $\Gamma$  have a form similar to (11).

The solutions for primary and adjoint plates enable us to express the variation of functional  $G$  explicitly in terms of variations of material and shape design functions of the rib. Consider first the variations of functionals  $G_1$  and  $G_2$  of (17). Then, following Dems and Mróz (1984, 1987) and taking into account conditions (95), it can be written

$$\delta G_i = \int (\Psi_{,Q} \cdot \delta Q + \Psi_{,q} \delta q + \Psi_{,f} \cdot \delta f + \Psi_{,u} \cdot \delta u) d\Omega + \int \Psi \delta \phi_n d\Gamma + \int (h_{,N_{ns}} \delta N_{ns} + h_{,N_n} \delta N_n + h_{,u_i} \delta u_i + h_{,u_n} \delta u_n + h \delta \phi_{,s}) dS_i \quad i = 1, 2 \quad (21)$$

where  $\delta$  denotes the local variation (i.e. for unperturbed domain) of state fields. Making use of (18)–(20), eqn (21) can be presented in the form

$$\delta G_i = \int (\mathbf{q}^a \cdot \delta \mathbf{Q} - \mathbf{Q}^a \cdot \delta \mathbf{q} + \mathbf{f}^a \cdot \delta \mathbf{u} + \Psi_{,f} \cdot \delta f) d\Omega + \int \Psi \delta \phi_n d\Gamma + \int [h_{,N_{ns}} (\delta N_{ns} - N_{ns} \delta \phi_s) + h_{,N_n} (\delta N_n - N_n \delta \phi_s) + h_{,u_i} (\delta u_i - u_{i,s} \delta \phi_s) + h_{,u_n} (\delta u_n - u_{n,s} \delta \phi_s) + (h \delta \phi_{,s})] dS_i \quad i = 1, 2. \quad (22)$$

Using now the virtual displacement principle for statically admissible stress field  $\mathbf{Q}^a$  and the kinematically compatible displacement field  $\delta \mathbf{u}$  as well as the virtual stress principle for fields  $\delta \mathbf{Q}$  and  $\mathbf{q}^a$ , the first integral on the right-hand side of eqn (22) can be rewritten in the form

$$\int (\dots) d\Omega_i = \int (\mathbf{u}^a + \Psi_{,f}) \cdot \delta f d\Omega + \int (\delta \mathbf{T} \cdot \mathbf{u}^a - \mathbf{T}^a \cdot \delta \mathbf{u}) dS_i + \int (\delta \mathbf{T} \cdot \mathbf{u}^a - \mathbf{T}^a \cdot \delta \mathbf{u}) d\Gamma \quad i = 1, 2. \quad (23)$$

The local variations  $\delta \mathbf{T}$  and  $\delta \mathbf{u}$  appearing in eqn (23) can be now replaced by the total variations  $\overset{\circ}{\delta} \mathbf{T}$  and  $\overset{\circ}{\delta} \mathbf{u}$  which take into account not only any variation for unperturbed domain  $\Omega$ , but also the variation due to shape modifications of  $\Omega_i$ ,  $S$  and  $\Gamma$ . Since we have (cf. Dems and Mróz, 1984)

$$\begin{aligned} \overset{\circ}{\delta} \mathbf{u} &= \delta \mathbf{u} + \mathbf{u}_{,k} \delta \phi_k \\ \overset{\circ}{\delta} \mathbf{T} &= \overset{\circ}{\delta} \mathbf{Q} \cdot \mathbf{n} + \mathbf{Q} \cdot \overset{\circ}{\delta} \mathbf{n} = \delta \mathbf{T} + \mathbf{Q}_{,k} \cdot \mathbf{n} \delta \phi_k - \mathbf{Q} \cdot \mathbf{t} (K \delta \phi_s + \delta \phi_{n,s}) \end{aligned} \quad (24)$$

where  $\delta \phi_k$  ( $k = 1, 2$ ) denote the components of transformation field  $\delta \phi$  with respect to fixed Cartesian coordinate system  $(x_1, x_2)$ ,  $K$  is the boundary curvature and  $\mathbf{Q}$  is treated as a symmetric tensor of generalized stresses, then eqn (23) can be reduced to the form

$$\int (\dots) d\Omega_i = \int (\mathbf{u}^a + \Psi_{,i}) \cdot \delta \mathbf{f} d\Omega_i + \int [\mathbf{u}^a \cdot (\delta \mathbf{T} - \mathbf{T}_{,i} \delta \phi_i) - \mathbf{T}^a \cdot (\delta \mathbf{u} - \mathbf{u}_{,i} \delta \phi_i) + (\mathbf{T} \cdot \mathbf{u}^a \delta \phi_i)_{,i}] dS_i + \int [\mathbf{u}^a \cdot (\delta \mathbf{T} - \mathbf{T}_{,i} \delta \phi_i) - \mathbf{T}^a \cdot (\delta \mathbf{u} - \mathbf{u}_{,i} \delta \phi_i) + (\mathbf{T} \cdot \mathbf{u}^a \delta \phi_i)_{,i} + (\mathbf{f} \cdot \mathbf{u}^a - \mathbf{Q} \cdot \mathbf{q}^a - \mathbf{T} \cdot \mathbf{u}^a \mathbf{K} + \mathbf{T} \cdot \mathbf{u}^a_{,n} + \mathbf{T}^a \cdot \mathbf{u}_{,n}) \delta \phi_n] d\Gamma \quad i = 1, 2. \quad (25)$$

Since the boundary tractions and displacements are specified with respect to the local coordinate system  $(\mathbf{n}, \mathbf{t})$  [cf. eqns (1), (2)], then it is more convenient, instead of their total variations with respect to a fixed coordinate system, to consider their corotational variations with respect to coordinate system  $(\mathbf{n}, \mathbf{t})$  moving together with transformed boundary  $\Gamma$  or  $S$ , cf. Dems and Mróz (1987), i.e. the variations which do not take into account the rotation of the coordinate system during the transformation process. In other words, if we denote by  $N_n, N_n$  and  $N_n^*, N_n^*$  the components of surface tractions in initial and actual configurations, then the corotational variations of surface traction are equal to differences  $N_n^* - N_n$  and  $N_n^* - N_n$ , respectively. Since these variations were denoted by  $\delta N_n, \delta N_n$  and  $\delta u_i, \delta u_i$ , we can write the following relationships between the total and corotational variations of surface tractions and displacements

$$\begin{aligned} \delta N_n &= \delta N_n - N_n(K\delta\phi_i + \delta\phi_{n,i}), & \delta N_n &= \delta N_n + N_n(K\delta\phi_i + \delta\phi_{n,i}) \\ \delta u_i &= \delta u_i - u_n(K\delta\phi_i + \delta\phi_{n,i}), & \delta u_i &= \delta u_i + u_n(K\delta\phi_i + \delta\phi_{n,i}). \end{aligned} \quad (26)$$

In view of (25) and (26), the sum of variations of functionals  $G_1$  and  $G_2$  defined over domain  $\Omega = \Omega_1 \cup \Omega_2$ , by using eqn (22), can be expressed as follows

$$\begin{aligned} \delta G_1 + \delta G_2 &= \int \{ (h_{,N_n} + u_n^a)(\delta \tilde{N}_n - \tilde{N}_n, \delta \phi_i) + (h_{,N_n} + u_n^a)(\delta \tilde{N}_n - \tilde{N}_n, \delta \phi_i) \} dS_i \\ &+ \int \{ [\delta N_n] u_i^a + [\delta N_n] u_n^a - [N_n^a] \delta u_i - [N_n^a] \delta u_n + ([N_n^a] u_{i,n} + [N_n^a] u_{n,i}) \\ &- [N_{n,i}] u_i^a - [N_{n,i}] u_n^a \delta \phi_i + ([f_i] u_i^a + [f_n] u_n^a - [N_n] u_i^a - [N_{n,i} v_n^a]) \\ &- [N_n] u_i^a K - [N_n] u_n^a K + [N_n u_{,n}^a] + [N_n^a u_{,n}^a] + [N_n^a u_{n,n}^a] + [\Psi] \delta \phi_n \\ &+ ([N_n] u_i^a - [N_n] u_i^a + [N_n^a] u_n - [N_n^a] u_n) \delta \phi_{n,i} \} d\Gamma + ([N_n] u_i^a + [N_n] u_n^a) \delta \phi_i \Big|_A^B \end{aligned} \quad (27)$$

where  $\Big|_A^B$  denotes the difference of enclosed quantities at points B and A calculated along  $\Gamma$ , and it was assumed that the surface tractions and displacements along external boundary  $S$  of primary and adjoint plates are continuous at points A and B.

The variation of functional  $G_r$  within rib domain can be calculated by following the analysis presented in Dems and Mróz (1987). Let us note that the ribs within primary and adjoint plates are treated as plane arches loaded by the jumps of plate internal forces  $[\mathbf{T}]$  and  $[\mathbf{T}^a]$  along  $\Gamma$ . Using the concept of corotational variations, the first variation of  $G_r$  can be expressed as follows (cf. Dems and Mróz, 1987)

$$\delta G_r = \int [\Phi_{,\Sigma} \cdot \delta \Sigma + \Phi_{,z} \cdot \delta z + \Phi(\delta \phi_{,i} - K \delta \phi_n) + \Phi_{,\gamma_i} (\delta \gamma_i + \gamma_{i,n} \delta \phi_i)] d\Gamma \quad (28)$$

where  $\delta \gamma_i$  denotes the local variation of stiffness function  $\gamma_i$  for unperturbed arch shape, and total variation  $\delta \gamma_i = \delta \gamma_i + \gamma_{i,n} \delta \phi_i$ . Using now eqns (14), (15) and (18)-(20), eqn (28) can be rewritten in the form



$$\delta G_r = \int [\boldsymbol{\varepsilon}^a \cdot \delta \boldsymbol{\Sigma} - \boldsymbol{\Sigma}^a \cdot \delta \boldsymbol{\varepsilon} + \Phi(\delta \phi_{,ss} - K \delta \phi_n)] d\Gamma + \int [\Phi_{,\gamma_i} - (\boldsymbol{\varepsilon}^a - \Phi_{,\boldsymbol{\Sigma}}) \cdot \mathbf{L}^i] (\delta \bar{\gamma}_i + \gamma_{i,s} \delta \phi_s) d\Gamma. \quad (29)$$

Eliminating now corotational variations  $\delta \boldsymbol{\Sigma}$ ,  $\delta \boldsymbol{\varepsilon}$  and  $\delta v$  by means of virtual stress and displacement equations derived by Dems and Mróz (1987), after some transformations, the first variation of the functional  $G_r$  is finally expressed as follows

$$\begin{aligned} \delta G_r = & \int (-[\delta N_{ns}] u_s^a - [\delta N_n] u_n^a + [N_{ns}^a] \delta u_s + [N_n^a] \delta u_n) d\Gamma \\ & + \int \{ ([N_{ns,s}] u_s^a + [N_{ns}] u_{n,s}^a - [N_{ns}^a] u_{,ss} - [N_n^a] u_{n,s}) \delta \phi_s + [(N^a u_n + N u_n^a - M_{,s} u_s^a - M^a_{,ss})_{,ss} \\ & - (N^a \varepsilon + M^a_{,x} + M_{,s} \theta^a + M^a_{,s} \theta - [N_{ns}] u_s^a - [N_n] u_n^a + \Phi) K] \delta \phi_n + [\Phi_{,\gamma_i} - (\boldsymbol{\varepsilon}^a - \Phi_{,\boldsymbol{\Sigma}}) \cdot \mathbf{L}^i] \cdot \delta \bar{\gamma}_i \} d\Gamma \\ & + [(N^a u_{,ss} - N_{,s} u_s^a + M^a_{,ss} u_{n,ss} + M_{,s} \theta^a + M^a_{,x} + \Phi) \delta \phi_s + (N^a u_n + N u_n^a - M^a_{,s} u_s - M_{,ss} u_s^a) \delta \phi_{n,s} \\ & - (N^a u_n + N u_n^a - M^a_{,ss} u_s - M_{,ss} u_s^a)_{,s} \delta \phi_n] \Big|_{\Lambda^r}^{\Gamma}. \end{aligned} \quad (30)$$

In writing eqn (30), we assumed that either the free end of the rib does not carry the concentrated load or the end is rigidly supported, and used the conditions (11).

Equations (27) and (30) enable us to determine the variation of functional  $G$ . Adding (17) and (30) and eliminating the jumps of internal forces along  $\Gamma$  within primary and adjoint plates by means of arch equilibrium equations, the first variation of functional  $G$  with respect to shape and material variations of a rib is expressed as follows

$$\begin{aligned} \delta G = & \int (\mathbf{u}^a + \boldsymbol{\Psi}_{,t}) \cdot \delta \mathbf{f} d\Omega + \int [(h_{,N_n} + u_n^a)(\delta \tilde{N}_{ns} - \tilde{N}_{ns,s} \delta \phi_s) + (h_{,N_n} + u_n^a)(\delta \tilde{N}_n - \tilde{N}_{n,s} \delta \phi_s)] dS_r \\ & + \int \{ [J_s] u_s^a + [J_n] u_n^a - [N_s] u_{,ss}^a + [N_{ns}^a] + [N_n^a] + N \theta^a_{,s} + N^a \theta_{,s} - (M_{,s} \varepsilon^a)_{,s} \\ & - (M^a_{,s} \varepsilon)_{,s} - (N^a \varepsilon + M^a_{,x}) K + [\boldsymbol{\Psi}] - \Phi K \} \delta \phi_n d\Gamma + \int [\Phi_{,\gamma_i} - (\boldsymbol{\varepsilon}^a - \Phi_{,\boldsymbol{\Sigma}}) \cdot \mathbf{L}^i] \cdot \delta \bar{\gamma}_i d\Gamma \\ & + [(NK u_n^a + N^a u_{,ss} + (M_{,s} u_n^a)_{,s} + M^a_{,ss} u_{n,ss} + M^a_{,x} + \Phi) \delta \phi_s + (M_{,s} \varepsilon^a + M^a_{,ss} \varepsilon - N \theta^a - N^a \theta) \delta \phi_n \\ & + (N^a u_n + N u_n^a - M^a_{,ss} u_s - M_{,ss} u_s^a) \delta \phi_{n,s}] \Big|_{\Lambda^r}^{\Gamma} = \delta G_\phi + \delta G_\gamma \end{aligned} \quad (31)$$

where  $\delta G_\phi$  and  $\delta G_\gamma$  denote the variations of  $G$  due to shape and material variations of a rib, respectively.

Thus, the variation of  $G$  is expressed explicitly in terms of variations of shape and material functions of a rib, as well as the solutions for primary and adjoint plates. The second integral on the right-hand side of eqn (31) expresses the variation of  $G$  due to variation of boundary conditions along the outer edge of plate, whereas the last term in square brackets expresses variation of  $G$  due to shape variation at both ends of rib within primary plate. When either the rib is closed or its ends are perpendicular to the outer edge of plate, then this term vanishes. Note furthermore, that the nonvanishing tangential variation  $\delta \phi$ , on  $S$  can be limited to these parts of  $S$  which are penetrated by the rib ends.

### 3.2. Sensitivity analysis for complementary and potential energies

Consider now a particular case when the functional  $G$  coincides with the complementary or potential energies of the plate and derive their first variations associated with

shape and stiffness variations. Such functionals occur in problems of global compliance or stiffness design.

Let us consider first the complementary energy of a disk stiffened with the rib which is closed or perpendicular to its outer edge. This energy equals

$$\Pi_\sigma = \int W(\mathbf{Q}) \, d\Omega + \int \tilde{W}(\boldsymbol{\Sigma}, \gamma) \, d\Gamma \tag{32}$$

where  $W$  and  $\tilde{W}$  denote the specific stress energies per unit area of a plate and unit length of a rib, respectively. Comparing (32) with (16), we easily observe that

$$\Psi = W \quad \text{in } \Omega, \quad h = 0 \quad \text{on } S, \quad \Phi = \tilde{W} \quad \text{along } \Gamma \tag{33}$$

and then, according to (18, 19), the adjoint plate has to satisfy the following boundary conditions

$$\tilde{\mathbf{T}}^a = 0 \quad \text{on } S_r, \quad \tilde{\mathbf{u}}^a = 0 \quad \text{on } S_u \tag{34}$$

with the imposed initial fields

$$\begin{aligned} \mathbf{f}^a = 0, \quad \mathbf{q}^a = W_{,\mathbf{Q}} = \mathbf{q}, \quad \mathbf{Q}^a = 0 \quad \text{in } \Omega \\ \boldsymbol{\varepsilon}^a = \tilde{W}_{,\boldsymbol{\Sigma}} = \boldsymbol{\varepsilon}, \quad \boldsymbol{\Sigma}^a = 0 \quad \text{along } \Gamma. \end{aligned} \tag{35}$$

Thus, the state fields within the adjoint plate are

$$\begin{aligned} \mathbf{u}^a = \mathbf{u}, \quad \mathbf{q}^a = \mathbf{q}, \quad \mathbf{Q}^a = 0 \quad \text{in } \Omega \\ \mathbf{v}^a = \mathbf{v}, \quad \boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}, \quad \boldsymbol{\Sigma}^a = 0 \quad \text{along } \Gamma \end{aligned} \tag{36}$$

and the first variation of  $\Pi_\sigma$ , in view of (31), equals

$$\begin{aligned} \delta\Pi_\sigma = \int \mathbf{u} \cdot \delta\mathbf{f} \, d\Omega + \int [u_n(\delta\tilde{N}_n - \tilde{N}_n, \delta\phi_n) + u_n(\delta N_n - \tilde{N}_n, \delta\phi_n)] \, dS_r \\ + \int \{ [\mathbf{f}] \cdot \mathbf{u} - [N_s] \boldsymbol{\varepsilon}_s + N\theta_s - (M_s, \boldsymbol{\varepsilon})_s + [W] - \tilde{W}K \} \delta\phi_n \, d\Gamma + \int \tilde{W}_{,\gamma_i} \cdot \delta\gamma_i \, d\Gamma. \end{aligned} \tag{37}$$

Assume now that functional  $G$  coincides with potential energy, that is

$$G = \Pi_u = \int [U(\mathbf{q}) - \mathbf{f} \cdot \mathbf{u}] \, d\Omega - \int \tilde{\mathbf{T}} \cdot \mathbf{u} \, dS_r + \int \tilde{U}(\boldsymbol{\varepsilon}, \gamma) \, d\Gamma \tag{38}$$

where  $U$  and  $\tilde{U}$  denote the specific strain energies. Comparing (38) with (16), we have

$$\Psi = U - \mathbf{f} \cdot \mathbf{u} \quad \text{in } \Omega, \quad h = 0 \quad \text{on } S_u, \quad h = -\tilde{\mathbf{T}} \cdot \mathbf{u} \quad \text{on } S_r, \quad \Phi = \tilde{U} \quad \text{along } \Gamma \tag{39}$$

and then, according to (18, 19), the adjoint plate is subject to the following boundary conditions

$$\tilde{\mathbf{T}}^a = h_{,u} = -\tilde{\mathbf{T}} \quad \text{on } S_r, \quad \tilde{\mathbf{u}}^a = 0 \quad \text{on } S_u \tag{40}$$

and the imposed fields of initial stresses and body forces

$$\begin{aligned} \mathbf{f}^u &= -\mathbf{f}, \quad \mathbf{q}^u = 0, \quad \mathbf{Q}^u = U_{,q} = \mathbf{Q} \quad \text{in } \Omega \\ \boldsymbol{\varepsilon}^u &= 0, \quad \boldsymbol{\Sigma}^u = \tilde{U}_{,i} = \boldsymbol{\Sigma} \quad \text{along } \Gamma. \end{aligned} \tag{41}$$

Thus, the state fields within the adjoint plate are

$$\begin{aligned} \mathbf{u}^u &= 0, \quad \mathbf{q}^u = 0, \quad \mathbf{Q}^u = -\mathbf{Q} \\ \mathbf{v}^u &= 0, \quad \boldsymbol{\varepsilon}^u = 0, \quad \boldsymbol{\Sigma}^u = -\boldsymbol{\Sigma} \end{aligned} \tag{42}$$

and the first variation of  $\Pi_u$ , in view of (31), equals

$$\begin{aligned} \delta \Pi_u &= - \int \mathbf{u} \cdot \delta \mathbf{f} \, d\Omega - \int [u_s(\delta \tilde{N}_{ss} - \tilde{N}_{ss}, \delta \phi_s) + u_n(\delta \tilde{N}_n - N_{n}, \delta \phi_n)] \, dS_\Gamma \\ &+ \int \{ -[[N_{ns}\varepsilon_{ns}] - [[N_n\varepsilon_n] - N\theta_{,i} + (M_{,s}\varepsilon)_{,s} + (N\varepsilon + M_x)K + [[U] - [[\mathbf{f}] \cdot \mathbf{u} \\ &\quad - \tilde{U}K] \delta \phi_n \, d\Gamma + \int \tilde{U}_{,i} \delta \gamma_i \, d\Gamma. \end{aligned} \tag{43}$$

Noting that  $U + W = \mathbf{Q} \cdot \mathbf{q}$  and  $\tilde{U} + \tilde{W} = \boldsymbol{\Sigma} \cdot \boldsymbol{\varepsilon}$ , it is easy to prove that  $\delta \Pi_u = -\delta \Pi_\sigma$ .

#### 4. SENSITIVITY ANALYSIS FOR PLATES IN FLEXURE

In this section we shall consider the case of bending of a plate shown in Fig. 3. The problem of optimal design of discrete stiffeners in plates was treated by Samsonov (1978). The present analysis, however, differs from that in Samsonov (1978). The optimal distribution of circular ribs in plastic plates was determined by Kozłowski and Mróz (1969). Optimization of densely stiffened plate was analysed in Rozvany *et al.* (1982, 1987) by assuming the equivalent orthotropic model.

The plate is subjected to a transverse load  $p$ , whereas either generalized tractions  $\tilde{\mathbf{R}}$  or displacements  $\tilde{\mathbf{w}}$  are specified on  $S$ . The rib within plate domain is simulated by a plane arch  $\Gamma$  with free or supported ends A and B, subjected to bending in  $\mathbf{t}$ ,  $\mathbf{b}$ -plane and torsion in  $\mathbf{n}$ ,  $\mathbf{b}$ -plane. Denote by  $\mathbf{v}$ ,  $\boldsymbol{\varepsilon}$ ,  $\boldsymbol{\Sigma}$  the generalized displacement, strain and stress fields specified along the rib axis, where now

$$\mathbf{v} = \begin{Bmatrix} v \\ \theta \end{Bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} x_b \\ x_t \end{Bmatrix}, \quad \boldsymbol{\Sigma} = \begin{Bmatrix} M_b \\ M_t \end{Bmatrix}. \tag{44}$$

Here  $v$  denotes the deflection of a rib in  $b$ -direction, and  $\theta$  is the angle of cross-section rotation along  $t$ -axis,  $x_b$  and  $M_b$  denote the bending curvature and moment, whereas  $x_t$  and  $M_t$  are the torsion and cross-sectional twisting moment. The relations between the generalized strains and displacements have the form

$$x_b = \vartheta_{,s} + K\theta, \quad x_t = \theta_{,s} - K\vartheta \tag{45}$$

where  $\vartheta = v_{,s}$  is the angle of cross-section rotation along  $\mathbf{n}$ -axis. The generalized displacements, strains and stresses within subdomains  $\Omega_1$  and  $\Omega_2$  of the plate are denoted by  $\mathbf{w}^1, \mathbf{w}^2, \mathbf{x}^1, \mathbf{x}^2$  and  $\mathbf{M}^1, \mathbf{M}^2$ , respectively.

It is obvious that the following kinematic relations between the generalized displacements and strains of the plate and rib hold along  $\Gamma$

$$\begin{aligned} v &= w^1 = w^2, \quad \vartheta = v_{,s} = w^1_{,ss} = w^2_{,ss}, \quad \theta = -w^1_{,sn} = -w^2_{,sn} \\ x_b &= w_{,sss} - Kw_{,sn} = -x_t^1 = -x_t^2, \quad x_t = -(w_{,sn} + Kw_{,ss}) = x_m^1 = x_m^2. \end{aligned} \tag{46}$$

The generalized stresses within a rib are related to the jump of internal forces within plate domains by the equilibrium equations of the form

$$T_{,s} - \llbracket Q \rrbracket = 0, \quad M_{b,s} + M_t K + T = 0, \quad M_{t,s} - M_b K - \llbracket M_n \rrbracket = 0 \tag{47}$$

where  $\llbracket M_n \rrbracket$  and  $\llbracket Q \rrbracket = \llbracket V + M_{n,s} \rrbracket$  denote the jumps of plate bending moment and generalized shear force along  $\Gamma$ , and  $T$  denotes the cross-sectional shear force within the rib.

We assume, similarly as previously, that generalized stress-strain relations within plate and rib domains are given in the general non-linear form (13). The incremental forms of stress-strain relations are expressed similarly to (14).

4.1. *Sensitivity analysis for an arbitrary functional*

Similarly as in Section 3, consider now the functional

$$G = \int \Psi(\mathbf{M}, \boldsymbol{x}, p, w) \, d\Omega + \int h(M_n, Q, w_{,n}, w) \, dS + \int \Phi(\boldsymbol{\Sigma}, \boldsymbol{\varepsilon}, \gamma) \, d\Gamma. \tag{48}$$

Rewriting the functional (48) in the form

$$G = G_1 + G_2 + G_r = \left( \int \Psi \, d\Omega_1 + \int h \, dS_1 \right) + \left( \int \Psi \, d\Omega_2 + \int h \, dS_2 \right) + \int \Phi \, d\Gamma \tag{49}$$

its first variation equals

$$\delta G = \delta G_1 + \delta G_2 + \delta G_r. \tag{50}$$

Following the analysis presented in Section 3, we introduce now the adjoint stiffened plate of the same shape as the primary one and subjected to the same kind of boundary conditions which are specified as follows

$$\tilde{w}^a = -h_{,Q}, \quad \tilde{w}^a_{,n} = h_{,M_n} \quad \text{on } S_u, \quad \tilde{Q}^a = h_{,w}, \quad \tilde{M}^a = -h_{,w_{,n}} \quad \text{on } S_r \tag{51}$$

where  $S_u$  and  $S_r$  are the supported and loaded parts of plate outer edge, respectively. Furthermore, the adjoint plate is subject to imposed fields of initial strain and stress specified by

$$\begin{aligned} \boldsymbol{x}^{ai} &= \Psi_{,M}, \quad \mathbf{M}^{ai} = \Psi_{,r} \quad \text{within } \Omega = \Omega_1 \cup \Omega_2 \\ \boldsymbol{\varepsilon}^{ai} &= \Phi_{,\boldsymbol{\Sigma}}, \quad \boldsymbol{\Sigma}^{ai} = \Phi_{,r} \quad \text{along } \Gamma \end{aligned} \tag{52}$$

and is loaded by the transverse pressure

$$p^a = \Psi_{,w} \quad \text{within } \Omega. \tag{53}$$

The stress field  $\mathbf{M}^a$  and  $\boldsymbol{\Sigma}^a$  within domain  $\Omega$  and along  $\Gamma$  are related to the strain fields  $\boldsymbol{x}^a$  and  $\boldsymbol{\varepsilon}^a$  by the linear relations, which are similar to (20). Obviously, the fields  $\mathbf{M}^a$  and  $\boldsymbol{\Sigma}^a$  satisfy the equilibrium conditions, while  $\boldsymbol{x}^a$  and  $\boldsymbol{\varepsilon}^a$  follow from the displacement fields  $w^a$  and  $v^a$ . The kinematic relations along  $\Gamma$  have a form similar to (46), while the static conditions are similar to those expressed by eqns (47).

Using the solutions for primary and adjoint plates we can express the variation of  $G$  explicitly in terms of material and shape variations of the plate rib. Following the analysis presented in Dems and Mróz (1989a), derive first the variation of functionals  $G_1$  and  $G_2$  of (48). Thus, we can write

$$\delta G_i = \int (\Psi_{,m} \cdot \delta M + \Psi_{,x} \cdot \delta x + \Psi_{,p} \delta p + \Psi_{,w} \delta w) d\Omega_i + \int \Psi \delta \phi_n d\Gamma + \int [h_{,m_n} (\delta M_n - M_{n,}, \delta \phi_s) + h_{,Q} (\delta Q - Q_{,}, \delta \phi_s) + h_{,w_n} (\delta w_n - w_{,n_s} \delta \phi_s) + h_{,w} (\delta w - w_{,s} \delta \phi_s)] dS_i + h \delta \phi_s |_{\Lambda^i}^B \quad i = 1, 2. \quad (54)$$

Using now (52), (53) and constitutive equations for primary and adjoint plates and applying the virtual displacement and stress principles, the first integral on the right-hand side of eqn (54) can be rewritten as follows

$$\int (\dots) d\Omega_i = \int (x^a \cdot \delta M - M^a \cdot \delta x + \Psi_{,p} \delta p + p^a \delta w) d\Omega_i = \int (w^a + \Psi_{,p}) \delta p d\Omega_i + \int (\delta V w^a - \delta M_{in} w^a_{,i} - V^a \delta w + M^a_{in} \delta w_{,i}) dS_i + \int (\delta V w^a - \delta M_{in} w^a_{,i} - V^a \delta w + M^a_{in} \delta w_{,i}) d\Gamma \quad (55)$$

where  $M_{in} = M_{i,j} n_j$  and  $V = M_{i,j} n_j$ .

Similarly as in Dems and Mróz (1989a), we can now eliminate the local variations  $\delta M_{in}$  and  $\delta V$  in eqn (55) by means of corotational variations of generalized boundary forces  $M_n$  and  $Q$ . Thus, after transforming of eqn (55) and substituting it into eqn (54), the sum of variations of functionals  $G_1$  and  $G_2$  equals

$$\begin{aligned} \delta G_1 + \delta G_2 = & \int (w^a + \Psi_{,p}) \delta p d\Omega + \int [(h_{M_n} - w^a_{,n}) (\delta M_n - M_{n,}, \delta \phi_s) + (h_{,Q} + w^a) (\delta Q - Q_{,}, \delta \phi_s) \\ & + (h_{,w_n} + M^a_{n}) (\delta w_n - w_{,n_s} \delta \phi_s) + (h_{,w} - Q^a) (\delta w - w_{,s} \delta \phi_s)] dS \\ & + \{ [M^a_{ns}] \delta w - [\delta M_{ns}] w^a - ([M_{ns}] w^a_{,s} + [M^a_{ns}] w_{,s}) \delta \phi_s \} |_{\Lambda^i}^B + \int \{ [p] w^a - [M_s] x^a_{,s} \\ & + 2 [M_{ns}] x^a_{,ns} + [Q] w^a_{,n} + [Q^a] w_{,n} + [M^a_{n} x_n] - ([Q] w^a - [M_n] w^a_{,n}) K - ([M_n] w^a_{,s})_{,s} \\ & - ([M^a_{n}] w_{,s})_{,s} \delta \phi_n + [[M_{n,s}] w^a_{,n} - [Q_{,s}] w^a - [M^a_{n}] w_{,ns} + [Q^a] w_{,s}] \delta \phi_s + [\delta Q] w^a \\ & - [\delta M_n] w^a_{,n} - [Q^a] \delta w + [M^a_{n}] \delta w_{,n} \} d\Gamma + \{ [M^a_{ns}] \delta w - [\delta M_{ns}] w^a + ([Q] w^a - [M_n] w^a_{,n} \\ & - [M_{ns}] w^a_{,s} - [M^a_{ns}] w_{,s}) \delta \phi_s + ([M_n] w^a_{,s} - [M_{ns}] w^a_{,n} + [M^a_{n}] w_{,s} - [M^a_{ns}] w_{,n}) \delta \phi_n \} |_{\Lambda^i}^B. \quad (56) \end{aligned}$$

Consider now the variation of functional  $G_r$  of (47) defined within rib domain. This variation can be obtained in the similar way as that presented in Dems and Mróz (1987) and Section 3. Thus,  $\delta G_r$  can be written first in the form similar to (28). Next, after using (51), (52) and the linear generalized stress-strain relation within the rib of adjoint plate, it can be retransformed to eqn (29) where now the generalized strain and stress have the components defined by eqn (44). To eliminate the variations  $\delta \Sigma$ ,  $\delta \varepsilon$  and  $\delta v$  in eqn (29) we have now to formulate the virtual stress and displacement equations for an arch of varying shape subjected to bending and torsion. To do this, let us denote the generalized load acting on the rib by

$$\mu = \begin{Bmatrix} q \\ m \end{Bmatrix} = \begin{Bmatrix} -[Q] \\ -[M_n] \end{Bmatrix} \quad (57)$$

and consider the identity following from (45), (47) and (57)

$$\int \Sigma \cdot \varepsilon d\Gamma - \int \mu \cdot v d\Gamma - (T v + M_t \theta + M_b \vartheta) |_{\Lambda^i}^B = 0. \quad (58)$$

The first variation of eqn (58) is expressed as follows :

$$\int (\delta \Sigma \cdot \varepsilon + \Sigma \cdot \delta \varepsilon - \delta \mu \cdot \mathbf{v} - \mu \cdot \delta \mathbf{v}) \, d\Gamma + \int (\Sigma \cdot \varepsilon - \mu \cdot \mathbf{v}) (\delta \phi_{,s} - K \delta \phi_n) \, d\Gamma - [\delta(Tv) + \delta(M_i \theta) + \delta(M_b \vartheta)]|_{\Lambda}^{\Gamma}. \quad (59)$$

The virtual displacement equation can be obtained from (59) by eliminating the terms involving the variations of static fields, whereas the virtual stress equation requires one to eliminate the terms involving the variations of kinematic fields. Taking the variational form of equilibrium equations (47) and kinematic relations (45), and applying the results to eqn (59), we can write finally the following virtual displacement equation for primary static fields and adjoint kinematic fields

$$\int (\Sigma^a \cdot \delta \varepsilon - \mu^a \cdot \delta \mathbf{v}) \, d\Gamma = \int \{ [M_{b,x_b}^a + M_{i,x_i}^a - (M_{b,s}^a + M_i^a K) \vartheta K - (M_i^a \vartheta - M_b^a \theta)_{,s}] \delta \phi_n + (M_{b,x_b}^a + M_{i,x_i}^a - q^a v_{,s} - m^a \theta_{,s}) \delta \phi_s \} \, d\Gamma + [(M_i^a \vartheta - M_b^a \theta)_{,s} \delta \phi_n - (M_i^a \vartheta - M_b^a \theta) \delta \phi_{n,s} - (M_i^a \theta_{,s} + M_b^a \vartheta_{,s} + T^a v_{,s}) \delta \phi_s + M_b^a \delta \vartheta + M_i^a \delta \theta + T^a \delta v] |_{\Lambda}^{\Gamma}. \quad (60)$$

and the virtual stress equation

$$\int (\delta \Sigma \cdot \varepsilon^a - \delta \mu \cdot \mathbf{v}^a) \, d\Gamma = \int \{ [(M_i \vartheta^a - M_b \theta^a)_{,s} + (M_{b,s} + M_i K) \vartheta^a K - (qv^a + m\theta^a) K] \delta \phi_n + (M_{b,x_b}^a + M_{i,x_i}^a - q_{,s} v^a - m_{,s} \theta^a) \delta \phi_s \} \, d\Gamma + [-(M_i \vartheta^a - M_b \theta^a)_{,s} \delta \phi_n + (M_i \vartheta^a - M_b \theta^a) \delta \phi_{n,s} - (M_{b,s} \vartheta^a + M_{i,s} \theta^a + T_{,s} v^a) \delta \phi_s + \delta M_b \vartheta^a + \delta M_i \theta^a + \delta T v^a] |_{\Lambda}^{\Gamma}. \quad (61)$$

Substituting now eqns (60) and (61) into eqn (29), the first variation of functional  $G$ , for the case of bending and torsion of a rib can be written, after some transformations, in the form

$$\delta G_r = \int \{ \delta q v^a + \delta m \theta^a - q^a \delta v - m^a \delta \theta - (q_{,s} v^a + m_{,s} \theta^a - q^a v_{,s} - m^a \theta_{,s}) \delta \phi_s + \{ (M_i \vartheta^a - M_b \theta^a + M_i^a \vartheta - M_b^a \theta)_{,s} + [(M_{b,s} + M_i K) \vartheta^a + (M_{b,s} + M_i^a K) \vartheta - q v^a - m \theta^a - M_{b,x_b}^a - M_{i,x_i}^a - \Phi] K \} \delta \phi_n \} \, d\Gamma + \int [\Phi_{,s} - (\varepsilon^a - \Phi_{,s}) \cdot \mathbf{L}^s] \cdot \delta \gamma_s \, d\Gamma + [(M_i \vartheta^a - M_b \theta^a + M_i^a \vartheta - M_b^a \theta)_{,s} \delta \phi_n - (M_{b,s} \vartheta^a + M_{i,s} \theta^a + T_{,s} v^a - M_b^a \vartheta_{,s} - M_i^a \theta_{,s} - T^a v_{,s} - \Phi) \delta \phi_s + \delta M_b \vartheta^a + \delta M_i \theta^a + \delta T v^a - M_b^a \delta \vartheta - M_i^a \delta \theta - T^a \delta v] |_{\Lambda}^{\Gamma}. \quad (62)$$

The expressions (56) and (62) enable us to determine the variation of the total functional  $G$ . Substituting these expressions into eqn (50) and using the conditions (46), (47) and (57), as well as noting that the rib shear force at points A and B equals (cf. Fig. 4b)

$$T = \llbracket M_{ns} \rrbracket^{\Gamma} + \llbracket M_{ns} \rrbracket^S \quad (63)$$

the first variation of  $G$  can be expressed as follows :

$$\begin{aligned}
 \delta G = & \int (w^a + \Psi_{,p}) \delta p \, d\Omega + \int [(h_{,M_n} - w_n^a)(\delta M_n - M_{n,s} \delta \phi_s) + (h_{,Q} + w^a)(\delta Q - Q_{,s} \delta \phi_s)] \, dS_T \\
 & + \int \{ [\Psi] - \Phi K + [p] w^a - [M_s] x_s^a - 2[M_{ns}] x_{ns}^a + [M_n^a x_n] + q\theta^a + (m\vartheta^a)_{,s} \\
 & + q^a \theta + (m^a \vartheta)_{,s} + (M_t \vartheta^a - M_b \theta^a + M_t^a \vartheta - M_b^a \theta)_{,ss} + [(M_{b,s} + M_t K) \vartheta^a + (M_{b,s}^a + M_t^a K) \vartheta \\
 & - M_b^a x_b - M_t^a x_t] K \} \delta \phi_n \, d\Gamma + \int [\Phi_{,\gamma_l} - (\varepsilon^a - \Phi_{,\Sigma}) \cdot L^l] \cdot \delta \gamma_l \, d\Gamma + [(\Phi + M_b^a \vartheta_{,s} + M_t^a \theta_s) \delta \phi_s \\
 & + (M_b \theta^a_{,s} - M_t \vartheta^a_{,s} + M_b^a \theta_{,s} - M_t^a \vartheta_{,s}) \delta \phi_n] |_{\Lambda^r}^{\text{B}^r}. \tag{64}
 \end{aligned}$$

In view of eqns (45) and (47) we can write the following equality valid for any static and kinematic fields

$$q\theta + (m\vartheta)_{,s} + (M_t \vartheta - M_b \theta)_{,ss} + (M_{b,s} + M_t K) \vartheta K = (M_t x_b)_{,s} - (M_b x_t)_{,s} + T x_t. \tag{65}$$

Using (65) in (64),  $\delta G$  can be finally expressed in the form

$$\begin{aligned}
 \delta G = & \int (w^a + \Psi_{,p}) \delta p \, d\Omega + \int [(h_{,M_n} - w_n^a)(\delta M_n - M_{n,s} \delta \phi_s) + (h_{,Q} + w^a)(\delta Q - Q_{,s} \delta \phi_s)] \, dS_T \\
 & + \int \{ [\Psi] - \Phi K + [p] w^a - [M_s] x_s^a - 2[M_{ns}] x_{ns}^a + [M_n^a x_n] + (M_t x_b^a - M_b x_t^a + M_t^a x_b - M_b^a x_t)_{,s} \\
 & + T x_t^a + T^a x_t - (M_b^a x_b + M_t^a x_t) K \} \delta \phi_n \, d\Gamma + \int [\Phi_{,\gamma_l} - (\varepsilon^a - \Phi_{,\Sigma}) \cdot L^l] \cdot \delta \gamma_l \, d\Gamma \\
 & + [(\Phi + M_b^a \vartheta_{,s} + M_t^a \theta_s) \delta \phi_s + (M_b \theta^a_{,s} - M_t \vartheta^a_{,s} + M_b^a \theta_{,s} - M_t^a \vartheta_{,s}) \delta \phi_n] |_{\Lambda^r}^{\text{B}^r} = \delta G_\phi + \delta G_\gamma. \tag{66}
 \end{aligned}$$

Thus, the variation of  $G$  is expressed explicitly in terms of variations of shape and material functions of a rib, as well as the solutions for primary and adjoint plates. The second integral on the right-hand side of eqn (66) expresses the variation of  $G$  due to variation of boundary conditions along loaded part  $S_T$  of the outer edge of plate, whereas the last term in square brackets expresses the variation of  $G$  due to shape variation at rib ends within primary plate. When the rib is closed then this term vanishes. Note, furthermore, that the nonvanishing tangential variation  $\delta \phi$ , on  $S$  can only be limited to these parts of  $S$  which are penetrated by rib ends.

#### 4.2. Sensitivity analysis for complementary and potential energies

Derive now the first variation of complementary and potential energies of plate associated with shape and stiffness variations. Assume then that the rib is closed or perpendicular to the outer edge of plate and consider first the complementary energy

$$\Pi_\sigma = \int W(\mathbf{M}) \, d\Omega + \int \tilde{W}(\Sigma, \gamma_l) \, d\Gamma \tag{67}$$

where  $W$  and  $\tilde{W}$  denote the specific stress energies of plate and rib. Comparing (67) with (48) we have

$$\Psi = W \quad \text{within } \Omega, \quad h = 0 \quad \text{on } S, \quad \Phi = \tilde{W} \quad \text{along } \Gamma \tag{68}$$

and then, according to (51), (52), the adjoint plate has to satisfy the homogeneous set of boundary conditions and is subjected to the imposed fields of initial curvatures

$$x^u = W, M = x \quad \text{within } \Omega, \quad \varepsilon^u = \tilde{W}, \Sigma = \varepsilon \quad \text{along } \Gamma. \tag{69}$$

Thus, the state fields within the adjoint plate are

$$\begin{aligned} w^u &= w, \quad x^u = x, \quad M^u = 0 \quad \text{within } \Omega \\ v^u &= v, \quad \varepsilon^u = \varepsilon, \quad \Sigma^u = 0 \quad \text{along } \Gamma \end{aligned} \tag{70}$$

and the first variation of  $\Pi_u$ , in view of (66), equals

$$\begin{aligned} \delta \Pi_u &= \int w \delta p \, d\Omega + \int [w(\delta Q - Q, \delta \phi_v) - w, n(\delta M_n - M_n, \delta \phi_v)] \, dS_r + \int \{ \llbracket W \rrbracket - \tilde{W} K + \llbracket p \rrbracket w \\ &\quad - \llbracket M_n \rrbracket, r_v - 2 \llbracket M_n \rrbracket, r_{nv} + (M_{t, r_b} - M_{b, r_t}), v + T, r_t \} \delta \phi_n \, d\Gamma + \int \tilde{W}, \gamma_l \delta \gamma_l \, d\Gamma. \end{aligned} \tag{71}$$

Note that since the rib is closed or perpendicular to the outer plate edge, then the boundary terms at points A and B on the right-hand side of eqn (66) vanish.

Assume now that the functional  $G$  coincides with the potential energy, that is

$$G = \Pi_u = \int [U(r) - pw] \, d\Omega - \int (Qw - M_n w, n) \, dS_r + \int \tilde{U}(\varepsilon, \gamma_l) \, d\Gamma \tag{72}$$

where  $U$  and  $\tilde{U}$  denote the specific strain energies. Comparing (72) with (48), we can easily observe that

$$\begin{aligned} \Psi &= U - pw \quad \text{within } \Omega, \quad h = 0 \quad \text{on } S_u, \quad h = Qw - M_n w, n \quad \text{on } S_r \\ \Phi &= \tilde{U} \quad \text{along } \Gamma \end{aligned} \tag{73}$$

and then, according to (51), (52), the adjoint plate is subjected to the following set of boundary conditions

$$w^u = 0 \quad \text{on } S_u, \quad Q^u = -Q, \quad M_n^u = -M_n \quad \text{on } S_r \tag{74}$$

and the imposed fields of initial generalized stresses

$$M^u = U, r = M \quad \text{within } \Omega, \quad \Sigma^u = \tilde{U}, \varepsilon = \Sigma \quad \text{along } \Gamma. \tag{75}$$

Moreover, the adjoint plate is loaded by the lateral pressure that, in view of (53), equals

$$p^u = \Psi, w = -p \quad \text{within } \Omega. \tag{76}$$

The state fields within the adjoint plate are then

$$w^u = 0, \quad x^u = 0, \quad M^u = -M, \quad v^u = 0, \quad \varepsilon^u = 0, \quad \Sigma^u = -\Sigma \tag{77}$$

and the first variation of  $\Pi_u$ , in view of (66), takes the form

$$\begin{aligned} \delta \Pi_u &= - \int w \delta p \, d\Omega + \int [w, n(\delta M_n - M_n, \delta \phi_v) - w(\delta Q - Q, \delta \phi_v)] \, dS_r + \int \{ \llbracket U \rrbracket - \llbracket p \rrbracket w - \tilde{U} K \\ &\quad - \llbracket M_n, r_n \rrbracket - (M_{t, r_b} - M_{b, r_t}), v - T, r_t + (M_{b, r_b} + M_{t, r_t}) K \} \delta \phi_n \, d\Gamma + \int \tilde{U}, \gamma_l \delta \gamma_l \, d\Gamma. \end{aligned} \tag{78}$$



Since  $U + W = \mathbf{M} \cdot \boldsymbol{\varepsilon}$  and  $\tilde{U} + \tilde{W} = \boldsymbol{\Sigma} \cdot \boldsymbol{\varepsilon}$  it is easy to prove that  $\delta \Pi_u = -\delta \Pi_r$ .

### 5. OPTIMALITY CONDITIONS IN OPTIMAL DESIGN OF RIBS

The typical optimal design problem involves minimization of the cost functional of a structure subject to the global constraint imposed on generalized stresses, strains or displacements. This global constraint can be expressed in the general form similar to (16) or (48). The other geometrical constraints, which can be additionally imposed, will not be considered here. Note that the constraints imposed on local values of stress, strain or displacement can easily be converted to the global form (16) or (48) by using the Dirac delta function. Similarly, any constraint imposed on maximum values of stress, strain or displacement components can also be expressed in the global form, as it was shown, for instance, in Mróz *et al.* (1985) or Dems and Mróz (1989a).

In what follows, we assume that the cost of plate domain  $\Omega$  is fixed and only the cost of a rib can vary due to change of its shape and cross-sectional properties. Thus, the objective function can be expressed in the form

$$C = \int c(\gamma) dS \Rightarrow \min_{\gamma, \phi} \quad (79)$$

where  $c$  is a specific material cost of a rib. The first variation of  $C$  now equals

$$\delta C = \delta C_\gamma + \delta C_\phi = \int c_{,\gamma} \cdot \delta \gamma dS - \int c K \delta \phi_n dS + c \delta \phi_n \Big|_{\lambda^r} \quad (80)$$

where  $\delta c_\gamma$  and  $\delta c_\phi$  denote the cost variation due to change of material properties and shape of rib, respectively. When the rib is closed or perpendicular to the outer edge of plate, then the last term on the right-hand side of eqn (80) vanishes.

Assume now that the plate is subject to the global constraint of the form

$$G - G_o \leq 0 \quad (81)$$

where  $G$  is expressed by eqn (16) or (48) and  $G_o$  is a prescribed quantity. To derive the optimality conditions of problem (79), (81), we introduce the Lagrange functional

$$G' = G + \lambda(G - G_o + \alpha^2) \quad (82)$$

where  $\lambda$  denotes the Lagrange multiplier and  $\alpha$  is a slack function. The stationarity condition of functional  $G'$  yields the optimality condition

$$\delta C = -\lambda \delta G \quad (83)$$

with the switching and constraint conditions of the form

$$\lambda \alpha = 0, \quad \delta \lambda (G - G_o + \alpha^2) = 0. \quad (84)$$

The optimality condition (83) can be rewritten in the form

$$\delta C_\gamma = -\lambda \delta G_\gamma, \quad \delta C_\phi = -\lambda \delta G_\phi \quad (85)$$

where  $\delta C_\gamma$  and  $\delta C_\phi$  are expressed by eqn (80), while  $\delta G_\gamma$  and  $\delta G_\phi$  are expressed by eqn (31) for stretching of plate and eqn (66) for the case of its bending.

An alternative formulation of the optimal design problem would require the minimization (or maximization) of  $G$  with the upper bound set on the structural cost, that is

$$\min \text{ (or max) } G \text{ subject to } C - C_0 \leq 0. \tag{86}$$

Introducing now the functional

$$G' = G + \lambda(C - C_0 + \beta^2) \tag{87}$$

where  $\lambda$  and  $\beta$  denote the Lagrange multiplier and slack function, we can obtain the following set of optimality conditions

$$\delta G_\gamma = -\lambda \delta C_\gamma, \quad \delta G_\phi = -\lambda \delta C_\phi, \quad \lambda \beta = 0, \quad \delta \lambda (C - C_0 + \beta^2) = 0 \tag{88}$$

which are equivalent to the conditions (84), (85).

### 6. EXAMPLES

In this section, let us consider three simple examples which should illustrate the sensitivity analysis and optimal design for rib-stiffened plates.

*Example 1.* Consider a circular disk of radius  $r_e$  with a central hole of radius  $r_i$ . The disk is stiffened with a circular rib of radius  $R$  and cross-sectional area  $A$  (Fig. 5), and is loaded uniformly by pressures  $p_e$  and  $p_i$  acting within the disk plane along the outer and inner edges. The disk is made of linear-elastic material with elastic constants  $E$  and  $\nu$ . Due to geometrical and mechanical symmetry, the nonvanishing stress components within disk domain are radial and circumferential stresses  $N_r$  and  $N_\theta$ , whereas the disk rib is subjected to tension with the normal force  $N$ . The stress state within the disk domains  $\Omega_1$  and  $\Omega_2$  is expressed as follows

$$N_{rr} = \frac{A_i}{r^2} + B_i, \quad N_{\theta\theta} = -\frac{A_i}{r^2} + B_i \quad i = 1, 2 \tag{89}$$

where  $A_i$  and  $B_i$  denote some constants, whereas the cross-sectional normal force within the rib is related to the jump of radial stresses by the relation [cf. eqn (12)]

$$[[N_r]] = N_{r1}|_{r=R} - N_{r2}|_{r=R} = -\frac{N}{R} \tag{90}$$

The conditions of displacement continuity within disk domain yields the following relations between the generalized strains within disk and rib

$$\epsilon_{r1}|_{r=R} = \epsilon_{r2}|_{r=R} = \epsilon \tag{91}$$

where  $\epsilon_\theta$  denotes the circumferential strain component within  $\Omega_i$  and  $\epsilon$  is the unit elongation of the rib. In view of disk boundary conditions and eqns (89)–(91), the constants  $A_1, B_1,$

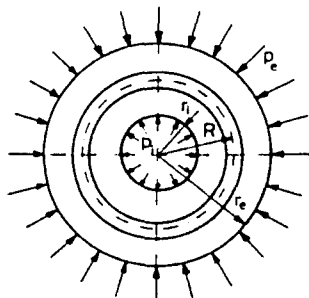


Fig. 5. Rib-stiffened circular disk loaded by internal and external pressures.

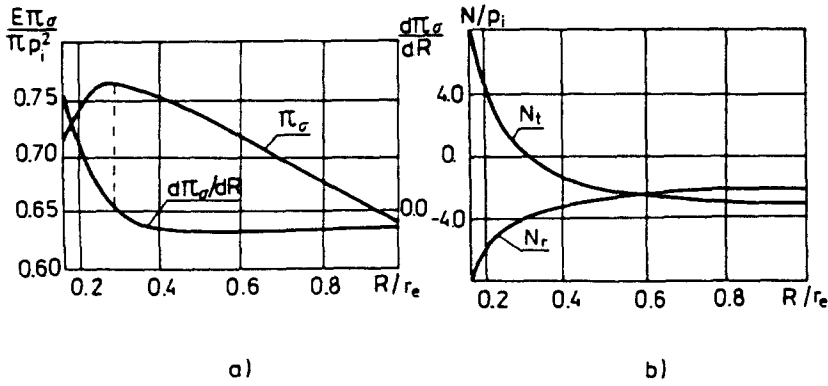


Fig. 6. Distribution of sensitivity of  $\Pi_\sigma$  versus varying rib radius (a) and generalized stresses for stationarity point of  $\Pi_\sigma$  (b) for constant rib cross-section.

$A_2$ ,  $B_2$  and the normal force  $N$  can be determined as the solutions of the following set of equations:

$$\begin{aligned} \frac{A_1}{r_1^2} + B_1 &= -p_1, & \frac{A_2}{r_e^2} + B_2 &= -p_e, & \frac{A_1 - A_2}{R^2} + B_1 - B_2 &= -\frac{N}{R}, \\ (1 - \nu)B_1 - (1 + \nu)\frac{A_1}{R^2} &= \frac{N}{A}, & (1 - \nu)B_2 - (1 + \nu)\frac{A_2}{R^2} &= \frac{N}{A}. \end{aligned} \quad (92)$$

Consider now the complementary energy of a disk that equals

$$\begin{aligned} \Pi_\sigma &= \int W \, d\Omega + \int \tilde{W} \, d\Gamma = \int_0^R (N_{r_1}^2 - 2\nu N_{r_1} N_{t_1} + N_{t_1}^2) r \, dr \\ &\quad + \int_0^R (N_{r_2}^2 - 2\nu N_{r_2} N_{t_2} + N_{t_2}^2) r \, dr + \frac{RN^2}{A} \end{aligned} \quad (93)$$

and determine its first variation with respect to the position of the rib and its cross-sectional area. In view of (37),  $\delta\Pi_\sigma$  is expressed in the form

$$\delta\Pi_\sigma = \int [([W] - \tilde{W}K)\delta\phi_n + \tilde{W}_{,\gamma_i}\delta\gamma_i] \, d\Gamma. \quad (94)$$

Assuming, for simplicity,  $\nu = 0$  and noting that  $\delta\phi_n = \delta R$  and  $\gamma_i = A$ , eqn (94) yields

$$\delta\Pi_\sigma = \frac{\pi}{E} \left\{ \left[ -[N_{r_1}(R) + N_{r_2}(R)]N + \frac{N^2}{A} \right] \delta R - \frac{RN^2}{A^2} \delta A \right\}. \quad (95)$$

In studying the first variation of  $\Pi_\sigma$  let us consider now three different kinds of variation of rib cross-sectional area  $A$ . Assume first the constant rib cross-section. Thus,  $\delta A = 0$  and the last term on the right-hand side of eqn (95) vanishes and the sensitivity of  $\Pi_\sigma$  with respect to  $R$  is expressed in the form

$$\frac{d\Pi_\sigma}{dR} = \frac{\pi}{E} N \left( \frac{N}{A} - \frac{A_1 + A_2}{R^2} - B_1 - B_2 \right) \quad (96)$$

where  $A_1$ ,  $A_2$ ,  $B_1$ ,  $B_2$  and  $N$  are the solutions of eqns (92). Figure 6a shows the graph of  $\Pi_\sigma$  and its sensitivity with respect to varying rib radius  $R$  for  $r_1 = 1$ ,  $r_e = 6$ ,  $A = A_0 = 2$  and

$p_c/p_i = 0.5$ . It is easy to note that  $\Pi_\sigma$  attains its minimum value for position of rib on the outer edge of disk, whereas for value of  $R$  satisfying the condition  $d\Pi_\sigma/dR = 0$ ,  $\Pi_\sigma$  attains its maximum value. For such position of the rib, the jump of normal stresses  $[N_r] = 0$  and the circumferential stress  $N_r = 0$ , as it is shown in Fig. 6b, and then the normal rib force vanishes.

Assume secondly that the rib cross-sectional area is inversely proportional to its radius, namely

$$A = \frac{r_1}{R} A_o, \quad \delta A = -\frac{r_1}{R^2} A_o \delta R = -\frac{A}{R} \delta R \tag{97}$$

where  $A_o$  is a prescribed quantity. Then, it follows from eqn (95) that

$$\frac{d\Pi_\sigma}{dR} = \frac{\pi}{E} N \left( \frac{2N}{A} - \frac{A_1 + A_2}{R^2} - B_1 - B_2 \right). \tag{98}$$

Figure 7a shows the graph of  $\Pi_\sigma$  and  $d\Pi_\sigma/dR$  with respect to varying  $R$  for the same values of  $r_i, r_c, A_o$  and  $p_c/p_i$  as in the previous case, while on Fig. 7b the graph of  $N_r, N_t$  for  $R$  satisfying the condition  $d\Pi_\sigma/dR = 0$  is plotted. We can observe that  $\Pi_\sigma$  attains a maximum for the same value of  $R$  as in previous case, whereas the minimum value of  $\Pi_\sigma$  corresponds to  $R = r_i$ .

Assume finally the quadratic relationship between  $A$  and  $R$ , namely

$$A = \hat{A}_o - k \left( R - \frac{r_i + r_c}{2} \right)^2, \quad \delta A = -2k \left( R - \frac{r_i + r_c}{2} \right) \delta R \tag{99}$$

where  $\hat{A}_o$  and  $k$  are the prescribed quantities. Then, in view of (95), the sensitivity of  $\Pi_\sigma$  takes the form

$$\frac{d\Pi_\sigma}{dR} = \frac{\pi}{E} N \left[ \frac{N}{A} - \frac{A_1 + A_2}{R^2} - B_1 - B_2 + 2k \frac{N}{A^2} R \left( R - \frac{r_i + r_c}{2} \right) \right]. \tag{100}$$

Assuming the same values of  $r_i, r_c$  and  $p_c/p_i$  as previously, and the values of  $\hat{A}_o$  and  $k$  satisfying the conditions  $A(r_i) = A(r_c) = A_o = 2$ , the graph of  $\Pi_\sigma$  and its sensitivity is shown

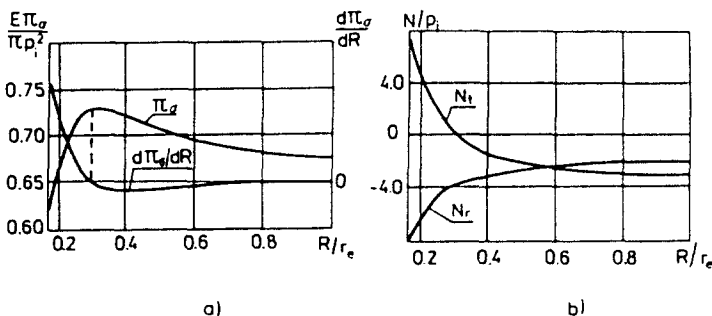


Fig. 7. Distribution of sensitivity of  $\Pi_\sigma$  versus varying rib radius (a) and generalized stresses for stationarity point of  $\Pi_\sigma$  (b) for rib cross-sectional area inversely proportional to its radius.

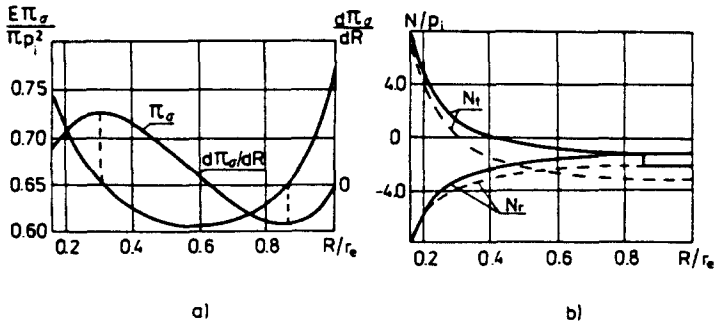


Fig. 8. Distribution of sensitivity of  $\Pi_\sigma$  versus varying rib radius (a) and generalized stresses for stationarity points of  $\Pi_\sigma$  (b) for quadratically varying rib cross-section.

in Fig. 8a. Now we observe, contrary to the previous cases, that besides the position of rib corresponding to maximization of  $\Pi_\sigma$ , the condition  $d\Pi_\sigma/dR = 0$  yields the value of  $R$  for which the complementary energy of a disk attains its global minimum. Figure 8b shows the graphs of radial and circumferential stresses within disk domain for both cases. For the case of global minimum of  $\Pi_\sigma$  we observe the nonvanishing normal force in the rib caused by the jump of normal stresses  $[[N_r]]$  on both sides of the rib.

*Example 2.* The next two examples are concerned with the bending of plates. Consider first a uniform circular plate of radius  $R$  and of bending stiffness  $a$  (Fig. 9). The plate is rigidly supported along the outer edge and is loaded by the uniform lateral pressure  $p$ . The material of plate and rib is assumed to be linearly elastic with elastic constants  $E$  and  $\nu$ . Due to symmetry of the problem, the nonvanishing stress and strain components within plate domain are the radial and circumferential bending moments  $M_r, M_\theta$  and curvatures  $\kappa_r, \kappa_\theta$ , whereas the rib is subjected to bending with the bending moment  $M_b$  and curvature  $\kappa_b$ . The moment  $M_b$  is related to the jump  $[[M_n]]$  by the equation, cf. eqn (47),

$$[[M_n]] = -M_b K = -a \kappa_b K. \tag{101}$$

The kinematic relations along the rib axis have the following form :

$$[[\kappa_n]] = -[[w_{,rr}]], \quad \kappa_r = Kw_{,rr}, \quad \kappa_{\theta s} = 0, \quad \kappa_b = -Kw_{,rr}, \quad \kappa_\theta = 0. \tag{102}$$

The deflection field for such plate was derived by Nash (1948) and it is expressed in the form

$$w_1 = -\frac{pr_c^4}{64D} \left\{ \left(\frac{r}{r_c}\right)^4 + 4 \ln \frac{r_c}{R} - 2\left(\frac{r}{R}\right)^2 + 1 - C \left[ \frac{1}{4} \frac{R^2 - r_c^2}{r_c^2} \left(\frac{r}{R}\right)^2 + \frac{1}{2} \ln \frac{r_c}{R} \right] \right\} \quad \text{for } 0 \leq r \leq R$$

$$w_2 = -\frac{pr_c^4}{64D} \left\{ \left(\frac{r}{r_c}\right)^4 + 4 \ln \frac{r_c}{R} - 1 - C \left[ \frac{1}{4} \left(\frac{r}{r_c}\right)^2 + \frac{1}{2} \ln \frac{r_c}{r} - \frac{1}{4} \right] \right\} \quad \text{for } R \leq r \leq r_c \tag{103}$$

where the constant  $C$  equals

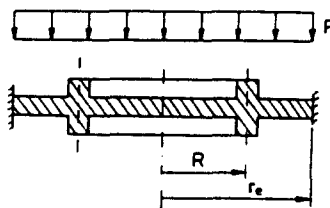


Fig. 9. Uniform circular plate stiffened by a ring.

$$C = \left\{ 8 \frac{R}{r_c} + 4 \left[ 1 - \left( \frac{R}{r_c} \right)^4 \right] \frac{a}{Dr_c} \right\} / \left\{ \frac{R}{r_c} + \frac{1}{2} \left[ 1 - \left( \frac{R}{r_c} \right)^2 \right] \frac{a}{Dr_c} \right\}. \tag{104}$$

Consider now the potential energy of the plate

$$\Pi_u = \int (U - pw) \, d\Omega + \int \tilde{U} \, d\Gamma = \pi \left\{ D \int_0^{r_c} (x_n^2 + 2v_n x_n x_s + x_s^2) r \, dr - \int_0^{r_c} \rho w r \, dr + a x_n^2 R \right\} \tag{105}$$

where  $D$  denotes the plate rigidity, and determines its first variation with respect to position of a rib and its bending stiffness. In view of (78),  $\delta \Pi_u$  can be written in the form

$$\delta \Pi_u = \int \left\{ \frac{1}{2} ([M_s]_{,x_s} - [M_n]_{,x_n}) + M_{b,r_n} K \right\} \delta \phi_n + \tilde{U}_{,r_l} \delta \gamma_l \, d\Gamma. \tag{106}$$

Expressing the static fields in terms of kinematic quantities and noting that  $\delta \phi_n = \delta R$  and  $\delta \gamma_l = \delta a$ , eqn (106) yields the following expression :

$$\delta \Pi_u = \pi DR \left\{ (-[x_n^2] + [x_n]_{,x_s}) \delta R - \frac{R}{a} [x_n]_{,x_s} \delta a \right\}. \tag{107}$$

In analysing the sensitivity of  $\Pi_u$  let us assume now two different kinds of variation of rib bending stiffness. When the rib stiffness is inversly proportional to its radius, then we have

$$a = \frac{r_c}{R} a_0, \quad \delta a = - \frac{r_c}{R^2} a_0 \delta R = - \frac{a}{R} \delta R \tag{108}$$

where  $a_0$  is a prescribed quantity. In view of (106) and (107), the sensitivity of  $\Pi_u$  with respect to  $R$  takes the form

$$\frac{d\Pi_u}{dR} = \pi DR [x_n] \{ 2x_s - (x_{n1} + x_{n2}) \}. \tag{109}$$

Assuming now constant value of rib stiffness, we have  $\delta a = 0$ , and then in view of (106), the sensitivity of  $\Pi_u$  equals

$$\frac{d\Pi_u}{dR} = \pi DR [x_n] \{ x_s - (x_{n1} + x_{n2}) \}. \tag{110}$$

Consider now the stationarity of  $\Pi_u$  with respect to the position of rib within plate domain. For the case of varying  $a$ , in view of eqn (109), the condition  $d\Pi_u/dR = 0$  yields the following optimality condition :

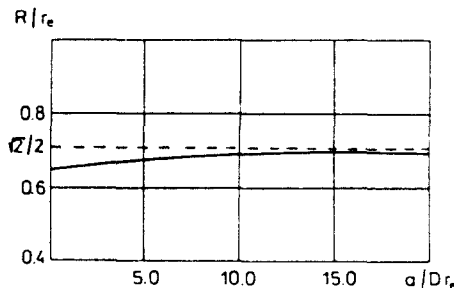


Fig. 10. Optimal value of rib radius versus plate stiffness parameter.

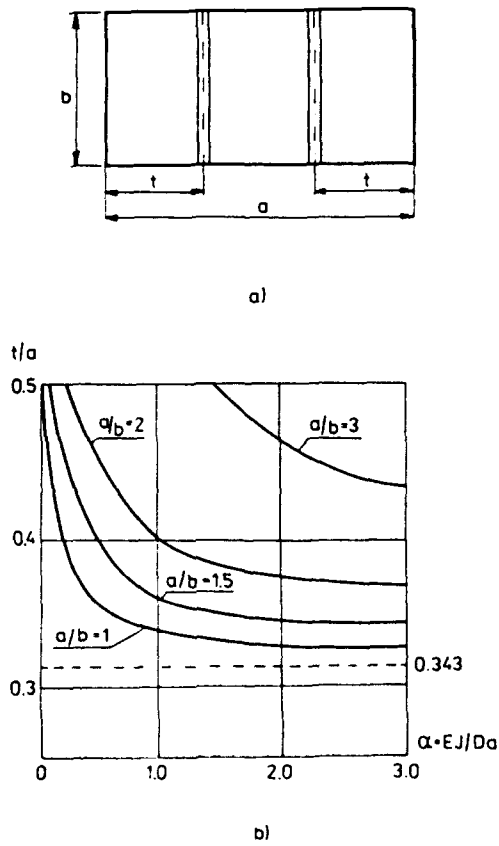


Fig. 11. Rectangular plate stiffened by two straight ribs (a) and optimal rib positions versus plate stiffness parameter (b).

$$w_{1,rr} + w_{2,rr} - \frac{2}{R} w_{,r} = 0. \tag{111}$$

Substituting eqns (103) and (104) into eqn (111), after some transformations, we have

$$\left[ 2 \left( \frac{R}{r_c} \right)^2 - 1 \right] \left\{ 4 \left( \frac{R}{r_c} \right)^2 + \left[ 1 - \left( \frac{R}{r_c} \right)^2 \right] \frac{a}{Dr_c} \right\} = 0 \tag{112}$$

from which the optimal value of  $R$  equals

$$R = \frac{\sqrt{2}}{2} r_c. \tag{113}$$

When, on the other hand, the rib stiffness is constant, in view of eqn (110), the condition  $d\Pi_u/dR = 0$  yields

$$w_{1,rr} + w_{2,rr} - \frac{1}{R} w_{,r} = 0. \tag{114}$$

Using once again eqns (103) and (104), we obtain the following optimality condition

$$\left( \frac{R}{r_c} \right)^3 + \left[ 1 - \left( \frac{R}{r_c} \right)^2 \right] \left[ 2 \left( \frac{R}{r_c} \right)^2 - 1 \right] \frac{a}{Dr_c} - 3 \frac{R}{r_c} = 0 \tag{115}$$

from which the optimal value of  $R$  can be determined. Figure 10 shows the graph of the optimal value  $R/r_c$ , with respect to varying parameter  $a/Dr_c$ . It is seen that for increasing value of  $a/Dr_c$  the optimal value of  $R$  tends asymptotically to the value  $\sqrt{2} r_c/2$ .

*Example 3.* This last example is concerned with the optimal position of two straight ribs reinforcing the rectangular plate of dimensions  $a \times b$  (Fig. 11a), which is loaded by the

lateral pressure  $p$  and is simply supported on its all external edges. Moreover, we assume that both ribs are positioned symmetrically at the distance  $t$  from the plate edges, and their torsional rigidity can be neglected.

Consider now the mean stiffness design, for which the optimality condition follows from the stationarity of potential energy of the plate. For the problem considered, in view of (78), the first variation of potential energy can be expressed as follows :

$$d\Pi_u = \int \left[ (M_{b,r_t})_{,s} - T_{,r_t} \right] d\Gamma \delta t = 0. \tag{116}$$

Using (46) and (47) and integrating by parts eqn (116) we obtain the following form of optimality condition

$$\int \llbracket Q \rrbracket w_{,n} d\Gamma = 0 \tag{117}$$

where  $\llbracket Q \rrbracket$  denotes the jump of plate generalized shear forces on both sides of rib axis.

Assuming now uniform lateral pressure  $p$ , let us consider the approximate solution of the problem at hand. We assume namely, that the plate is connected with a rib at  $l$  points, where  $l$  tends to infinity. Thus, the optimality condition (117) can be replaced by its approximate form

$$\sum_{i=1}^l P_i w_{i,s} \Big|_{y=\eta_i} = 0 \tag{118}$$

where  $(t_i, \eta_i)$  are the Cartesian coordinate of the  $i$ th point and  $P_i$  denotes the interaction force between the plate and rib at this point. The forces  $P_i$  ( $i = 1, 2, \dots, l$ ) can be determined from the condition of compatibility of plate and rib deflections at points  $(t_i, \eta_i)$ .

The total deflection of plate is equal to the sum of deflection of uniform plate subjected to the lateral pressure  $p$ , which can be obtained from the Navier solution, cf. Timoshenko and Woinowsky-Krieger (1959), and deflection of plate caused by interaction forces between the plate and ribs. Thus, it can be written

$$w(x, y) = w^p(x, y) - \sum_{i=1}^l P_i w'_i(x, y) \tag{119}$$

where

$$w^p(x, y) = \frac{16p}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \quad m, n = 1, 3, 5, \dots$$

$$w'_i(x, y) = \frac{4}{\pi^3 ab D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left( \sin \frac{m\pi t}{a} + \sin \frac{m\pi(a-t)}{a} \right) \sin \frac{n\pi \eta_i}{b}}{\left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \tag{120}$$

The deflection of rib can be expressed in the form



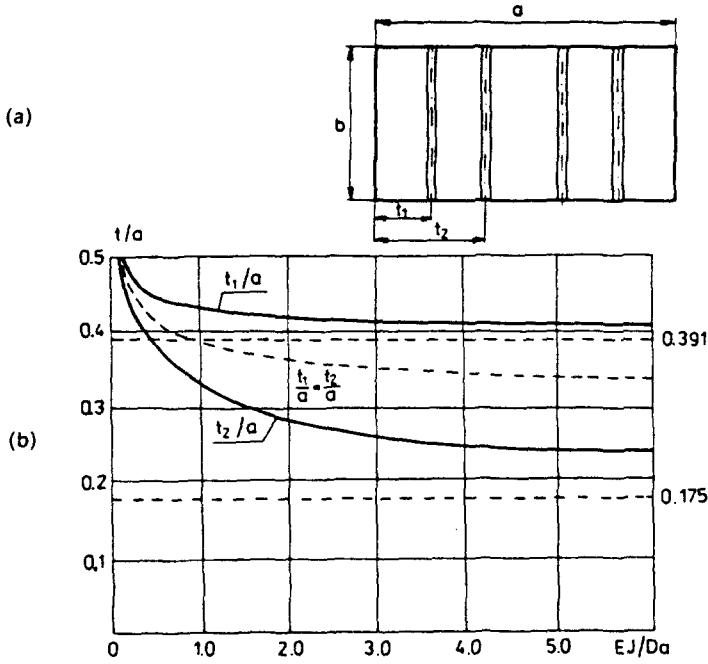


Fig. 12. Rectangular plate stiffened by four straight ribs (a) and optimal rib positions versus plate stiffness parameter (b).

$$v(y) = \sum_{i=1}^l P_i v_i(y) \tag{121}$$

where  $v_i$  denote the influence coefficients of simply supported beam which equal

$$v_i(y) = \frac{b^3}{6} \left[ \left( \frac{\eta_i}{b} \right) \left( 1 - \frac{\eta_i}{b} \right) \left( 2 - \frac{\eta_i}{b} \right) \frac{y}{b} - \left( 1 - \frac{\eta_i}{b} \right) \left( \frac{y}{b} \right)^3 \right] \quad \text{for } y \leq \eta_i$$

$$v_i(y) = \frac{b^3}{6EJ} \left\{ \left( \frac{\eta_i}{b} \right) \left[ 1 - \left( \frac{\eta_i}{b} \right)^2 \right] \left( 1 - \frac{y}{b} \right) - \frac{\eta_i}{b} \left( 1 - \frac{y}{b} \right)^3 \right\} \quad \text{for } y \geq \eta_i \tag{122}$$

where  $EJ$  denotes the bending stiffness of the rib.

Now, the interaction forces  $P_i$  can be determined from the condition

$$v(\eta_i) = w(t, \eta_i). \tag{123}$$

Substituting (119) and (121) into (123), we obtain the following set of equations,

$$\sum_{i=1}^l [w'_i(t, \eta_i) + v_i(\eta_i)] P_i = w''(t, \eta_i) \quad i = 1, 2, \dots, l \tag{124}$$

from which all  $P_i$  can be determined.

Knowing the forces  $P_i$ , the plate deflection  $w$  and its derivative  $w_{,x}$  are calculated from eqn (119), and next the results are substituted into optimality equation (118). Figure 11b shows the results of numerical solution of eqn (118) for  $l = 10$ ,  $\eta_i = i/11$ , where the nondimensional position of rib  $\bar{t} = t/a$  versus the parameter  $\alpha = EJ/Da$  is plotted. It is seen, that for small values of rib stiffness the ribs are positioned on the axis of symmetry of plate, whereas for increasing rib stiffness they are translated towards the plate edges tending to their terminal position specified by  $\bar{t} = 0.313$  for  $\alpha$  tending to infinity. A similar analysis for four ribs placed symmetrically provides an optimal solution illustrated in Fig. 12.

## 7. CONCLUDING REMARKS

The present paper provides a general variational method to treat problems of sensitivity analysis and optimal stiffener design by considering varying traction discontinuity lines. The sensitivities of arbitrary differentiable functionals are expressed explicitly in terms of state fields of primary and adjoint plates. These expressions can next be used in both analytical and numerical solutions for optimal design or identification problems. Numerical implementation through finite element method would require determination of both primary and adjoint displacement fields at nodal points along the rib and within the plate domain, so the respective discontinuities along rib line can be calculated. The study of various numerical approximation schemes is beyond the scope of this paper and will be discussed separately.

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